AUTOMODEL PROBLEM OF THE PENETRATION OF A SOLID BODY INTO THE GROUND

(AVTOMODEL'NAIA ZADACHA O PRONIKANII TVERDOGO TELA V GRUNT)

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Problems of penetration of a solid body into the ground (or about underground explosions) are solved when certain assumptions as to the properties of the investigated medium are made. In this work, such a schematic medium is an ideal (non viscous) and barotropic gas. The body in motion here is a circular cone with a constant velocity of penetration (with the angle of attack = 0).

1. Statement of problem. Uniform gas fills an infinite plane at rest at initial time $t = 0$; then it is separated by a cylinder whose radius increases proportionally with time. Let the velocity of the cylinder (i.e. the velocity of its radial increase) be equal to U. Let t = time and r = distance from a point to the axis of the cone (Euler's coordinate); $p(r, t)$, $\rho(r, t)$, $u(r, t)$ are respectively the pressure, the density and the velocity of the gas. We will designate the initial values of these quantities by an index 0.

Because of cylindrical symmetry, the equation of motion and continuity can be written as [1]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \qquad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{\rho u}{r} = 0 \tag{1.1}$$

The third equation (barotropy) which in our case takes the place of the energy equation, will have the form:

$$p - p_0 = nk^2(\rho^n - \rho_0^n) \tag{1.2}$$

where n is a dimensionless constant which we will consider to be different from zero and unity.

The first of the equations (1.1) can now be written as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + n^2k^2\rho^{n-2}\frac{\partial \rho}{\partial r} = 0 \qquad (1.3)$$

The determining dimensional parameters are r, t, U, ρ_0, p_0, k. Because between these parameters we have the dimensional relation

$$[U^2] = [p_0][\rho_0^{-1}], \quad [k^2] = [p_0][\rho_0^{-n}],$$

this motion is selfsimilar. Let us put $\alpha = \rho^{1-n}U^2$, $\beta = U$, determining parameters with independent dimensions. In the case considered, the only non-dimensional variable combination will be

$$\lambda = Utr^{-1} \qquad (1.4)$$

and for density, velocity and the dimensional constant in (1.3) we have

$$\rho = \alpha^{\frac{1}{1-n}} r^{-\frac{2}{1-n}} t^{\frac{2}{1-n}} y, \qquad u = rt^{-1}x, \quad n^2k^2 = \alpha a^2 \qquad (1.5)$$

where x and y are dimensionless quantities and are therefore functions of only one variable: a is a dimensionless constant. Substituting (1.5) into (1.3) and into the second equation of (1.1), we have

$$\lambda\left[(x-1)\frac{dx}{d\lambda} + a^2y^{n-2}\frac{dy}{d\lambda}\right] = x^2 - x + \frac{2a^2}{n-1}y^{n-1}$$
$$\lambda\left[y\frac{dx}{d\lambda} + (x-1)\frac{dy}{d\lambda}\right] = \frac{2n}{n-1}xy - \frac{2}{n-1}y \qquad (1.6)$$

Dividing the first equation by the second we have

$$\frac{dz}{dx} = \frac{z}{x}\,\frac{z + (n+1)x(1-x)/2a^2 - (1-x)/a^2}{z - (1-x)^2/2a^2} \qquad (1.7)$$

where we have substituted $z = y^{n-1}$. From (1.6) we find

$$\frac{d\lambda}{dx} = \frac{\lambda}{2x}\,\frac{z - (1-x)^2/a^2}{z - (1-x)^2/2a^2} \qquad (1.8)$$

If we succeed in integrating (1.7), then $x(\lambda)$ and $z(\lambda)$ are in quadrature. But before starting any further investigation of (1.7), we have to determine the initial and the boundary conditions with respect to the new variables. For the condition of rest ($u = 0$), variables x and z correspond to a straight line $x = 0$, or the z-axis. For the portions adjoining the piston we have $r = Ut$, $u = U$, it follows that $x = 1$. Hence $0 \leqslant x \leqslant 1$.

Now we have to consider the relationships for a strong (and a weak) discontinuity, considering that the waves are propagating into a gas that is in a state of rest. Designating the quantities beyond the jump by the index 1 we can write

$$\rho_0(u_0 - c) = \rho_1(u_1 - c), \qquad p_0 + \rho_0(u_0 - c)^2 = p_1 + \rho_1(u_1 - c)^2$$
$$p_1 - p_0 = nk^2(\rho_1{}^n - \rho_0{}^n) \tag{1.9}$$

considering that in front of the shock wave, (1.2) is valid where c is the velocity of the shock (or the sound) wave.

In selfsimilar motion $c = r/t$ = const. Substituting $c = r/t$ and the relations (1.5) into (1.9), and remembering that $u_0 = 0$ we will find

$$z_1 = \frac{n}{a^2}\frac{x_1(1 - x_1)}{1 - (1 - x_1)^n}$$

for $x_1 = 0$ we have $z_1 = a^{-2}$. As can be seen, this point corresponds to a weak shock. In this fashion strong and weak shocks in the variables x and z are described by a single curve

$$z_1 = \frac{n}{a^2}\frac{x(1 - x)}{1-(1 - x)^n} \tag{1.10}$$

which is analogous to a shock adiabatic curve. When a particle crosses the shock wave, then the corresponding point in the plane xz jumps from a straight line $x = 0$ onto curve (1.10) (see [1]).

Referring to (1.8) we shall establish that curves $z_2 = (1 - x)^2/a^2$, $z_3 = (1 - x)^2/2a^2$ are boundaries which divide regions of increasing (decreasing) parameter with respect to the increases in x. Above $z_2(x)$ and below $z_3(x)$ quantities x and λ increase simultaneously; in the region between them, λ decreases while x increases. It is easy to establish that curves $z_1(x)$ and $z_2(x)$ have only two common points (at $x = 0$ and $x = 1$), with the exception of $n = -1$ when the two curves coincide. Here z_1 is above z_2 for $n > -1$ and z_1 is below z_2 for $n < -1$.

We will now study the behavior of the integral curves of equation (1.7) in the plane (x, z) for $0 \leqslant x \leqslant 1$ and $z > 0$ (for $z \leqslant 0$ the problem has no meaning). At the points of intersection of the straight line $z = 0$ and parabola $z_4 = (n + 1)\,x(x - 1)/2a^2 + (1 - x)/a^2$, the integral curves have tangent lines parallel to the x-axis. At the points of intersection of the line $x = 0$ and parabola $z_3(x)$, these tangent lines are parallel to the z-axis. Points

$$O(0,0),\ A\left(0,\ \frac{1}{a^2}\right),\quad B(1,0), C\left(\frac{1}{n},\ \ \frac{1}{2a^2}\left(\frac{1-n}{n}\right)^2\right),\quad D(\infty,\infty)$$

are singular points of the equation. At points O, A, B we have a node and at C and D a saddle point.

The integral curves are shaped as shown in Figs. 1 and 2. The arrows indicate the direction of increase of parameter λ along the integral curve which corresponds to the motion towards the center of symmetry.

Corresponding to the state of rest are the $x = 0$ points of the integral curve. Point $O(\lambda = 0)$ corresponds to an infinite point of the flow $(r = \infty)$. For the particles adjoining the piston, we have $x = 1$, $r = Ut$, $\lambda = 1$ according to (1.4). Hence, with motion in the physical plane from the piston to infinity, λ decreases continuously from 1 to 0. However, as we can see from Fig. 1 for $n > -1$, the integral curves under consideration (the ones crossing line $x = 1$) cross parabola $z_2(x)$ on which λ reaches a minimum. Hence continuous transition is impossible. It follows that in these cases motion from line $x = 1$ to point O is possible only when somewhere between $x = 1$ and $z = z_2(x)$ a jump takes place, as a result of which the describing point lands on line $x = 0$. On the other hand, it has been shown that the points on the z-axis jump on to curve (1.10), which at $n > -1$ is crossed by the integral curves. The following picture can be drawn: during the motion along the physical plane from the piston to infinity, the describing point moves along the integral curve from the line $x = 1$ to its intersection with curve $z_1(x)$ at some point $x = x_1$, $z = z_1$, after which it jumps on to line $x = 0$. A shock wave is created. Thus for $n > -1$, the problem is reduced to integrating the system of equations (1.7) and (1.8) at boundary conditions

$$z = \frac{n}{a^2}\frac{x_1(1-x_1)}{1-(1-x_1)^n} \quad \text{for } x = x_1, \qquad \lambda = 1 \quad \text{for } x = 1 \qquad (1.11)$$
$$\lambda = U/c \quad \text{for } x = x_1$$

where c is the velocity of the shock wave, which can be expressed by x_1.

We have three conditions to determine two arbitrary constants from the integration of two equations of the first order and an unknown constant x_1 i.e. the problem has one solution. In view of $\lambda(x_1) < \lambda(1) = 1$, $U < C$ i.e. the velocity of the shock wave turns out to be higher than the velocity of the piston.

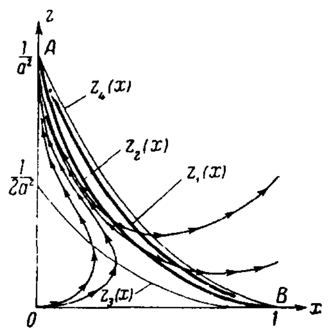

Fig. 1.

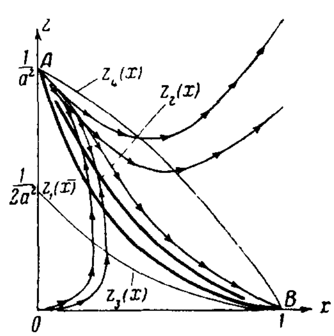

Fig. 2.

Let us now consider the cases with $n < -1$. Figure 2 shows that the integral curves crossing line $x = 1$ do not cross either $z_1(x)$ (which describes the jump) or curves $z_2(x)$, $z_3(x)$ on which λ reaches its extreme. All integral curves go through point $A(0_1, a^{-2})$, which, as has been shown, represents a shock wave in the xz-plane. And so, during the motion in a physical plane from the piston to infinity, the describing point moves along the integral curve from line $x = 1$ to the z-axis and further along the z-axis to point 0. A shock wave propagates throughout the gas. The problem now is to integrate the system of equations (1.7) and (1.8) with conditions

$$\lambda = 1 \quad \text{for } x = 1, \qquad \lambda = U/c_0 \quad \text{for } x = 0 \tag{1.12}$$

where c is the sound velocity in an undisturbed gas, a quantity which is fixed for all integral curves (with n fixed).

However, there is a limitation on U because $\lambda(0) < \lambda(1) = 1$, $U < c_0$. This condition becomes clear when we are reminded that cases with $n < -1$ are characteristic of the following: the curve of dependence of $\sigma = -p$ on $1/\rho$ (and therefore of p on deformation $\epsilon = 1 - \rho_0/\rho$) has a concavity upward and cannot be continued into the region of large deformations, as this curve has to have $\epsilon = 1$ as its vertical asymptote.

Therefore, if in the choice of n and k (i.e. in the approximation of a real curve) and a given velocity U, we get $U > c_0$ (or $U < c_0$ but of the same order) it will mean that the condensation is quite large and (1.2) does not hold. The general conclusion of this paragraph agrees with the known fact that if curve $p(\epsilon)$ is concave upward, then the discontinuous initial conditions cannot lead to a discontinuous solution. If, however, $p''(\epsilon) > 0$, then the initial conditions of flow of compression create a shock wave.

Let us consider the function

$$W = (\rho/\rho_0)^{\frac{1-n}{2}} = \lambda\sqrt{z}$$

considering (1.7) and (1.8) we can get

$$\frac{d\ln W}{dx} = \frac{1-n}{4a^2}\,\frac{1-x}{z-(1-x)^2/2a^2}$$

It is now clear that for our problem $W_x' > 0$ for $n < 1$ and $W_x' < 0$ for $n > 1$. As $\lambda'(x) > 0$, during the motion of the wave toward the piston, the density continuously increases.

Analogously we can show that $u_\lambda' > 0$. Hence it follows that as time goes by the velocity and the density of the gas particles continuously increase.

2. Let us now consider the case of $n = -1$, which is of practical importance as a large number of real media such as clay, sand and metals

have this property. Here the dependence $p(\epsilon)$ is linear over a large set of values of ϵ. For $n = -1$, the family of integral curves of equation (1.7) which cross the line $x = 1$ and satisfy the condition of $z > 0$, is given by a relation

$$z = \frac{(1-x)^2 + \sqrt{(1-x)^4 + b^2x^2}}{2a^2} \tag{2.1}$$

where b is an arbitrary constant of integration.

For $n = -1$ curve $z_1(x)$ (the curve of jumps) has the form $z_1 = (1-x)^2/2a^2$ and coincides with curve $z_2(x)$. Together they coincide with the integral curve (2.1) for which $b = 0$. And so all integral curves which cross line $x = 1$ go above $z_1(x)$ and $z_2(x)$, which corresponds to the propagation of a sound wave in the physical plane. The form of these integral curves is shown in Fig. 3. The direction of increase of λ is shown by arrows. Qualitatively this case is identical to the case with $n < -1$ of the preceding paragraph, and all remarks made there are also valid here. However, the fact that a shock wave cannot arise here is not an expected result, as with $p''(\epsilon) = 0$ initial discontinuities may either lead to a shock or to a motion without shock. Let us in fact consider this problem not only in cylindrical but also in planar (stroke on a rod) and spherical (spherical explosion) conditions. Corresponding to system (1.7) and (1.8), we have system

$$\frac{dz}{dx} = \frac{2}{\nu+1}\frac{z}{x}\frac{z - [1-(2-\nu)x + (1-\nu)x^2]/a^2}{z-(1-x)^2/(1+\nu)a^2} \tag{2.2}$$

$$\frac{dx}{d\lambda} = \frac{x}{\lambda}\frac{z-(1-x)^2/(1+\nu)a^2}{z-(1-x)^2/a^2} \tag{2.3}$$

where $\nu = 0$ for the planar case and $\nu = 2$ for a spherical case. For $\nu = 0$ we get from (2.2) and (2.3) $z = b^2x^2$ and $x = b_1{}^2\lambda$. It can be shown that

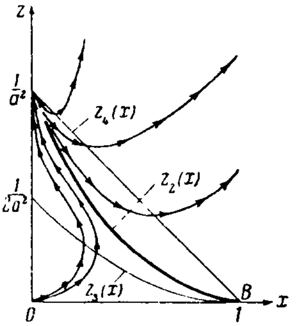

Fig. 3.

in this case a shock wave is created. Between the wave and the piston we have a region of constant flow. The problem is solved with conditions

(1.11). On the other hand for $\nu = 2$ a shock wave cannot be created but we have a sound wave, this can be shown by investigating the general solution of equation (2.2) at $\nu = 2$:

$$\sqrt{z}\,[z - (1 - x)^2 / a^2] = b^2 x$$

The problem is solved for conditions (1.12). Let us now return to the problem at hand. Substituting (2.1) into (1.8) we get

$$\ln \lambda = \frac{1}{2} \int \frac{\sqrt{(1 - x)^4 + b^2x^2} - (1 - x)^2}{x\sqrt{(1 - x)^4 + b^2x^2}}\, dx \tag{2.4}$$

and both conditions (1.12) reduce to one (put $\lambda_0 = U\rho_0 / k$):

$$\ln \lambda_0 = \frac{1}{2} \int_1^0 \frac{\sqrt{(1 - x)^4 + b^2x^2} - (1 - x)^2}{x\sqrt{(1 - x)^4 + b^2x^2}}\, dx \tag{2.5}$$

From this we find b.

For practical values of $U(\lambda_0 << 0.5)$ from (2.5) we can get an approximate formula $b = 2\sqrt{1 - 2\lambda_0^2} / \lambda_0^2$. Hence we can get a relation from (1.4), (1.5), and (2.1)

$$\frac{\rho}{\rho_0} = \frac{\lambda}{\lambda_0} \frac{\sqrt{2}}{\sqrt{(1 - x)^2 + \sqrt{(1 - x)^4 + b^2x^2}}}$$

It follows that for particles around the piston ($x = 1$, $\lambda = 1$) we have $\rho_0/\rho = \lambda_0\sqrt{b} / \sqrt{2}$. Substituting the approximate formula for b here, and neglecting all terms above the third degree in the exponential expansion of λ_0, we get that near the piston $\epsilon = 1 - \rho_0/\rho = 0.5\,\lambda_0^2$.

3. If curve $p(\epsilon)$ is concave downward throughout, i.e. $p''(\epsilon) > 0$, a shock wave is created and the following method of solution can be suggested. Let us assume (Fig. 4) that the dependency $p(\epsilon)$ is linear starting with some quite large values of ϵ. A shock wave is thus created such that the describing point on Fig. 4 heads from zero on to the linear part of the diagram. Between the wave and the piston (2.1) and (2.4) will be valid; only the boundary conditions will be changed. Let $\epsilon = \epsilon_1$ and $p = p_1$ directly behind the shock wave (Fig. 4). Also let

$$p_1 - p_0 = E_1\varepsilon_1 \quad \text{или} \quad p_1 - p_0 = k_1^2(\rho_0^{-1} - \rho_1^{-1})$$

Clearly $E_1 = k_1^2\rho_0$. Further, for the linear portion of $p(\epsilon)$ let

$$p - p_0 = E\varepsilon + \text{const}, \quad p - p_0 = k^2(\rho_0^{-1} - \rho^{-1}) + \text{const}, \quad E = k^2\rho_0$$

(obviously $E > E_1$). Then analogously to the derivation of curve (1.10) we can derive the equation of the ray OB of Fig. 4:

$$z_1 = (1 - x_1)^2 / a_1^2 \qquad (a_1 = k_1 / \rho_0 U)$$

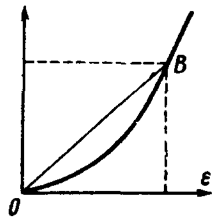

Fig. 4.

Qualitatively this case is identical with cases for $n > -1$ (Section 1), but here the constant k_1 is unknown (or E_1, a_1).

Using the dependence $p_1 - p_0 = E_1 \epsilon_1$, and taking into account that $ut = rx$, from the original relations (1.9) we can get the following two equalities.

$$x_1 = \varepsilon_1, \quad c = \sqrt{E_1 \rho_0^{-1}} = k_1 \rho_0^{-1}$$

(where c = velocity of the shock wave).

The following boundary conditions can be established

$$\lambda = 1 \quad \text{for } x = 1, \qquad \lambda = a_1^{-1} \quad \text{for } x = x_1$$
$$z = (1 - x_1)^2 / a_1^2 \qquad \text{for } x = x_1 \tag{3.1}$$

From Fig. 4, we can see that $E_1 / E = (\varepsilon_1 - \varepsilon_0) / \varepsilon_1 = a^2 / a_1^2$.

Now we can derive the dependence $a_1 = \sqrt{(x_1 - \varepsilon_0)\, x_1^{-1}} / \lambda_0$.

Using this, we can narrow down the first two conditions (3.1) to one

$$\ln \lambda_0 + \frac{1}{2} \ln \frac{x_1}{x_1 - \varepsilon_0} = \frac{1}{2} \int_1^{x_1} \frac{\sqrt{(1-x)^4 + b^2 x^2} - (1-x)^2}{x \sqrt{(1-x)^4 + b^2 x^2}}\, dx \tag{3.2}$$

The third condition will be

$$b = \frac{2\sqrt{\varepsilon_0}}{x_1 - \varepsilon_0} \frac{(1 - x_1)^2}{\sqrt{x_1}} \tag{3.3}$$

And so with given λ_0 and ϵ_0 we have to solve (3.2) and (3.3) simultaneously and substitute the value of b into (2.1) and (2.4). It is obvious that the constant b in this problem (with fixed ϵ_0) and the problem of Section 2 depend only upon λ_0 and not upon ρ_0, U or k separately. It is interesting to note that for practical values of $U(\lambda_0 \ll 0.5)$ condition (3.2) can be replaced by an approximation:

$$\lambda_0^2 \frac{x_1}{x_1 - \varepsilon_0} \frac{2 + \sqrt{4 + b^2}}{1 + \sqrt{1 + b^2 x_1^2}} = 1$$

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